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First Quantization of Mass and Charge

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The proper time is introduced as a parameter into the wave functions of relativistic quantum theory by first quantization of the mass. The classical limit is shown to be given by a recently developed canonical formulation of classical relativistic mechanics. The adjoint spinor is redefined with the help of a sign operator to remove a discrepancy between the classical and quantum actions in the behavior under time inversion. This results in positive energy densities for the Dirac theory. The inclusion of this sign operator into the definition of the probability current then removes negative probabilities from the theory. A five-dimensional formulation with first quantized charge is given.

1. INTRODUCTION

In classical relativistic mechanics the trajectory of a pointlike object is parameterized by the proper time of this object. The proper time is invariant under transformations of the Poincaré group. This corresponds to the role that time plays in classical nonrelativistic mechanics, where trajectories are parameterized by the time and, apart from translations, time is invariant under transformations of the Galilean group. Now in nonrelativistic quantum mechanics time is retained as a parameter, the time evolution of a single-particle wave function is described by the Schrödinger equation, and classical nonrelativistic mechanics is obtained in the limit $\hbar \to 0$. But there is no corresponding limit of relativistic quantum theory, since neither the Dirac equation nor the Klein–Gordon equation contains the proper time. Clearly, one might argue that the classical limit of nonrelativistic quantum mechanics and the inverse procedure, first quantization, are based on the Hamiltonian formulation of classical mechanics, whereas the Dirac and Klein–Gordon equations are covariant and no covariant Hamiltonian formulation of classical relativistic mechanics exists. But recently we argued (Hannibal, 1991)

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that a covariant Hamiltonian and canonical formalism indeed exist if one accepts that the existence is postulated independently from the existing Lagrangian formalism. We showed that the covariant classical action

$$S=\int d\tau\,[p_\mu(\tau)\dot{x}^\mu(\tau)-c^2M(p(\tau),x(\tau))] \tag{1}$$

with a covariant "mass function" $M(p, x)$ under free variation of $p(\tau)$ and $x(\tau)$ yields the canonical equations

$$\dot{x}^\mu=c^2\frac{\partial M}{\partial p_\mu} \tag{2a}$$

$$\dot{p}_\mu=c^2\frac{\partial M}{\partial x^\mu} \tag{2b}$$

which give the correct equations of motion if we postulate for a free particle

$$M(p,x)=c^{-1}(p_\mu\eta^{\mu\nu}p_\nu)^{1/2} \tag{3}$$

with Minkowskian metric $\eta^{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ and

$$M(p,x)=c^{-1}[(p_\mu-qA_\mu(x))g^{\mu\nu}(x)(p_\nu-qA_\nu(x))]^{1/2} \tag{4}$$

for a particle of charge q in external gravitational field $g^{\mu\nu}$ and electromagnetic field A_μ. We use the term "mass function" instead of "Hamiltonian" to distinguish it from the noncovariant energy function. In both cases the canonical equations (2) give the correct equations of motion for particles of any mass; due to $(d/d\tau)M(p(\tau),x(\tau))=0$ the rest mass becomes a constant of motion, and the parameter τ is fixed by equation (2a) to be the proper time. We argued that the formalism cannot be equivalent to the well-known Lagrangian formulation, for two reasons: First, the action (1) with mass function given by (3) or (4) does not contain a constant with physical dimension of mass; thus, it is valid for particles of any mass, whereas the Lagrangian contains the rest mass as a constant. Second, the canonical equations fix the parametrization of the trajectories to be the proper time, whereas it is not fixed by the Euler–Lagrange equations.

We now take this covariant canonical formulation of classical relativistic mechanics as a basis for a reanalysis of relativistic quantum mechanics. We ask: How does a relativistic quantum theory have to look in order to yield this theory as its classical limit?

2. GENERALIZED EQUATIONS OF MOTION

Obviously the wave functions of this new theory have to depend on the proper time τ in order to give τ-dependent expectation values $\langle x^\mu\rangle(\tau)$ of the coordinates for which then an analog to Ehrenfest's theorem should hold. Then the action and equations of motion should not contain the rest mass as a constant, this should be a relativistic invariant conserved quantity. Both requirements are satisfied if the wave functions depend on τ and in the action, Dirac, and Klein–Gordon equations the constant mass m_0 is replaced by the operator $-i\hbar c^{-2}\partial/\partial\tau$. By this procedure of first quantization of the mass we obtain generalized covariant equations for free particles that are

$$(i\hbar\gamma^\mu\partial_\mu + i\hbar c^{-1}\partial_\tau)\Psi(x, \tau) = 0 \tag{5}$$

instead of the Dirac (1928) equation and

$$(c^{-2}\partial_\tau^2 + \eta^{\mu\nu}\partial_\mu\,\partial_\nu)\Phi(x, \tau) = 0 \tag{6}$$

instead of the Klein (1926*b*)–Gordon (1926) equation. We write $\Psi(x, \tau)$ for Dirac spinors (spin 1/2) and $\Phi(x, \tau)$ for scalar wave functions (spin 0), and use ψ and ϕ for τ-independent functions. In both cases the separation ansatz

$$F(x, \tau) = \text{const} \cdot \exp(ic^2 m_0\tau/\hbar) \cdot f(x) \tag{7}$$

gives back the original equations. This resembles the change from the time-dependent to the time-independent Schrödinger equation. We now first analyze the classical limit of the generalized Dirac equation (5). The four-dimensional Fourier transform

$$\Psi(p, \tau) = \int d^4x\, e^{ip_\mu x^\mu/\hbar}\Psi(x, \tau) \tag{8}$$

obeys the equation

$$\partial_\tau\Psi(p, \tau) = (ic/\hbar)\gamma^\mu p_\mu \Psi(p, \tau) \tag{9}$$

which is solved, for given $\Psi(p, \tau_0)$, by

$$\Psi(p, \tau) = \exp[(ic/\hbar)\gamma^\mu p_\mu(\tau - \tau_0)] \cdot \Psi(p, \tau_0) \tag{10}$$

Using the properties of the Dirac γ-matrices, we have

$$\begin{aligned}\exp(ic/\hbar\gamma^\mu p_\mu\tau) &= [\exp(ic\tau\sqrt{p^2}/\hbar)](1 + \gamma^\mu p_\mu/\sqrt{p^2})/2\\ &\quad + [\exp(-ic\tau\sqrt{p^2}/\hbar)](1 - \gamma^\mu p_\mu/\sqrt{p^2})/2\end{aligned} \tag{11}$$

where we write $\sqrt{p^2} = \sqrt{p_\mu\eta^{\mu\nu}p_\nu}$ for short. This means that the operators

$$\hat{P}_\pm = (1 \pm \gamma^\mu p_\mu/\sqrt{p^2})/2 \tag{12}$$

are projectors onto states with positive ($\hat{P}_+$) or negative ($\hat{P}_-$) mass, with mass $m_\pm(p) = \pm c^{-1}\sqrt{p^2}$ and τ-evolution $\exp[ic^2 m_\pm(p)\tau/\hbar]$. The Dirac equation therefore states that particles have positive mass m_0. Solutions of the generalized equation in general do not have a definite mass; they will be linear combinations of arbitrary mass eigenstates. In a complete theory we hence need some generating mechanism that tells us which are the physically allowed masses.

For particles with positive mass we have

$$\hat{P}_-\Psi(p,\tau)=(1-\gamma^\mu p_\mu/\sqrt{p^2})/2\cdot\Psi(p,\tau)=0 \tag{13}$$

Hence all components Ψ_α of the spinor Ψ satisfy the scalar equation

$$\partial_\tau\Psi_\alpha(p,\tau)-(ic\sqrt{p^2}/\hbar)\Psi_\alpha(p,\tau)=0 \tag{14}$$

This scalar equation in momentum representation does not Fourier transform to a differential equation for $\Psi_\alpha(x,\tau)$, but is a pseudo-differential equation (Reed and Simon, 1975). We nevertheless take it also as the equation for scalar wave functions Φ because it is of first order in ∂_τ and its solutions have positive mass, whereas the generalized Klein–Gordon equation, which may be obtained as the square of (14), has solutions of both positive and negative mass. Moreover, we will later see that dimensional considerations favor this equation.

The τ-propagator $G_{\alpha\beta}(x,\tau)$ for the generalized Dirac equation (5), defined by

$$\Psi_\alpha(x,\tau)=\int d^4x'\,G_{\alpha\beta}(x-x',\tau-\tau')\Psi_\beta(x',\tau') \tag{15}$$

is given by

$$G_{\alpha\beta}(x,\tau)=\int\frac{d^4p}{(2\pi)^4}\left\{\left[\exp\frac{-i(p_\mu x^\mu-c\tau\sqrt{p^2})}{\hbar}\right]\hat{P}_+ + \left[\exp\frac{-i(p_\mu x^\mu+c\tau\sqrt{p^2})}{\hbar}\right]\hat{P}_-\right\} \tag{16}$$

The τ-propagator $G(x,\tau)$ for equation (14), defined by

$$\Phi(x,\tau)=\int d^4x'\,G(x-x',\tau-\tau')\Phi(x',\tau') \tag{17}$$

is given by

$$G(x,\tau)=\int\frac{d^4p}{(2\pi)^4}\exp\frac{-i(p_\mu x^\mu-c\tau\sqrt{p^2})}{\hbar} \tag{18}$$

We see that for positive mass in both cases the phase factors of $G(x-x',\tau-\tau')$ are precisely given by the classical action (1) for a trajectory with constant momentum $p(\tau)\equiv p$ and arbitrary path from x' at τ' to x at τ, divided by $\hbar$. This indicates that these equations indeed have the desired classical limit; this completely parallels the nonrelativistic case. The introduction of the τ-propagators eliminates the special role that time plays in the discussion of relativistic quantum physics. We consider this a step in the "reconciliation of quanta and relativity" desired by de Broglie (1939).

3. EHRENFEST'S THEOREM

We now look for the analog of Ehrenfest's (1927) theorem. The τ-dependent expectation values of the coordinates are defined by

$$\langle x^\mu\rangle_{\mathrm{D}}(\tau)=\left(\int d^4x\,\bar{\Psi}x^\mu\Psi\right)\bigg/\left(\int d^4x\,\bar{\Psi}\Psi\right)\tag{19}$$

for spinors, where $\bar{\Psi}$ is the Pauli (1936) adjoint spinor, and

$$\langle x^\mu\rangle_{\mathrm{KG}}(\tau)=\left(\int d^4x\,\Phi^*x^\mu\Phi\right)\bigg/\left(\int d^4x\,\Phi^*\Phi\right)\tag{20}$$

for scalar wave functions. For any operator $\hat{A}$ we have

$$(d/d\tau)\int d^4x\,\bar{\Psi}\hat{A}\Psi=(-ic/\hbar)\int d^4x\,\bar{\Psi}[\gamma^\mu p_\mu,\hat{A}]\Psi\tag{21}$$

and

$$(d/d\tau)\int d^4x\,\Phi^*\hat{A}\Phi=(-ic/\hbar)\int d^4x\,\Phi^*[\sqrt{p},\hat{A}]\Phi\tag{22}$$

Hence

$$\frac{d}{d\tau}\int d^4x\,\bar{\Psi}\Psi=\frac{d}{d\tau}\int d^4x\,\Phi^*\Phi=0\tag{23}$$

and we obtain in analogy to the nonrelativistic case

$$\frac{d}{d\tau}\langle\hat{A}\rangle=\frac{c^2}{i\hbar}\langle[M_+,\hat{A}]\rangle\tag{24}$$

In momentum representation we have $\hat{x}^\mu=(\hbar/i)\,\partial/\partial p_\mu$; hence

$$[\gamma^\mu p_\nu,\hat{x}^\mu]=i\hbar\gamma^\mu\tag{25}$$

and

$$[\sqrt{p^2},\hat{x}^\mu]=i\hbar\eta^{\mu\nu}p_\nu/\sqrt{p^2}\tag{26}$$

Since for positive mass

$$\bar{\Psi}\gamma^{\mu}\Psi=\bar{\Psi}(\eta^{\mu\nu}p_{\nu}/\sqrt{p^2})\,\Psi \tag{27}$$

we get in both cases

$$\frac{d}{d\tau}\langle x^{\mu}\rangle=\eta^{\mu\nu}\langle \hat{M}_{+}^{-1}p_{\nu}\rangle, \qquad \frac{d}{d\tau}\langle p_{\mu}\rangle=0 \tag{28}$$

with self-adjoint positive-mass operators

$$\hat{M}_{+\mathrm{D}}=c^{-1}\gamma^{\mu}p_{\mu}\hat{P}_{+}=c^{-1}\sqrt{p^2}\hat{P}_{+} \tag{29}$$

$$\hat{M}_{+\mathrm{KG}}=c^{-1}\sqrt{p^2} \tag{30}$$

We have thus derived a relativistic Ehrenfest theorem. We also conclude that

$$\frac{d}{d\tau}\langle \hat{M}_{+}\rangle=0 \tag{31}$$

Hence the mass is a conserved quantity, which is now a nontrivial statement. Thus, with respect to the equations of motion our relativistic quantum theory is consistent with the classical theory.

4. ACTIONS AND SYMMETRIES

But we have ignored a problem. Whereas $\int d^4x\,\Phi^*\Phi$ is always positive, $\int d^4x\,\bar{\Psi}\Psi$ is not necessarily, since we know from the solutions of the free Dirac equation with the Dirac representation of the γ-matrices (Bjorken and Drell, 1964) that for particle solutions with $p_0>0$ the upper components of Ψ dominate, and hence, $\int d^4x\,\bar{\Psi}\Psi$ will be positive; and for antiparticle solutions with $p_0<0$ the lower components dominate, and hence $\int d^4x\,\bar{\Psi}\Psi$ will be negative due to the definition of the adjoint spinor (Pauli, 1936),

$$\bar{\Psi}=\Psi^{*T}\gamma^0 \tag{32}$$

Thus, for linear combinations of particle and antiparticle solutions $\int d^4x\bar{\Psi}\Psi$ may become zero. The basic reason for this may be seen to be a question of the symmetries of the quantum theory. In the classical theory there were two transformations that interchanged particle and antiparticle solutions, namely the inversion of either time or proper time. The time inversion is a coordinate transformation defined by

$$x^{\mu\prime}=T^{\mu}{}_{\nu}x^{\nu}, \qquad p'_{\mu}=p_{\nu}T^{-1\nu}{}_{\mu}, \qquad T=\mathrm{diag}(-1,1,1,1) \tag{33}$$

First we note that the standard time inversion of the Dirac theory (Bjorken and Drell, 1964),

$$\psi'(Tx) = i\gamma^1\gamma^3\psi^*(x) \tag{34}$$

does not extend this transformation, since due to the additional complex conjugation, p_0 is not changed. But then there is an extension of the classical time inversion to our theory, given by Pauli (1936):

$$\begin{aligned} \Phi'(x', \tau) &= \Phi(x, \tau) \\ \Psi'(x', \tau) &= \gamma^0\gamma^5\Psi(x, \tau) \end{aligned} \tag{35}$$

Under this transformation the equations of motion (5) and (14) are invariant, but $\int d^4x\, \bar{\Psi}\Psi$ changes sign, $\int d^4x\, \Phi^*\Phi$ does not. This also applies to the actions. The action for scalar wave functions given by

$$S(\Phi) = \mathscr{R}(2\pi\hbar)^{-4} \int_{\tau_0}^{\tau_1} d\tau \int d^4p\, \Phi^*(p, \tau)(i\hbar\partial_\tau - ic\sqrt{p^2})\Phi(p, \tau) \tag{36}$$

($\mathscr{R}$ denotes the real part) yields equation (14) from the variational principle; it is invariant under the transformation (35). On the other hand, the action for spinors given by

$$S(\Psi) = \mathscr{R} \int_{\tau_0}^{\tau_1} d\tau \int d^4x\, \bar{\Psi}(x, \tau)(i\hbar\partial_\tau + i\hbar c\gamma^\mu\partial_\mu)\Psi(x, \tau) \tag{37}$$

yields the generalized Dirac equation under variation, but is not invariant under this transformation. This seems to be the reason why the transformation (35) was not adopted for the standard theory. But as a consequence there is no complex linear time inversion symmetry in this formulation of the Dirac theory. The symmetry properties of the classical and quantum actions as well as scalar and spinor actions are different.

We note that both Φ and Ψ have the same physical dimension of $(\text{length})^{-2}$, whereas in standard Klein–Gordon theory the dimension of ϕ involves the mass. This aspect of unification led us to choose the action (36) for spin-0 particles. We are consistent with standard theory without proper time if the modulus of constant in the separation ansatz (7) is chosen to be $1/(\Delta\tau)^{1/2}$, where $\Delta\tau = \tau_1 - \tau_0$ is the interval of τ integration in the action; we then obtain τ-independent actions.

Now the essential point of this paper is that the discrepancy in the symmetry properties and other well-known problems in the interpretation of relativistic quantum theory may be removed by a redefinition of the adjoint spinor. At first it seems that there is no freedom left in the definition of the adjoint spinor. We have to require (Pauli, 1936) that $\bar{\Psi}\Psi$ is a relativistic invariant real scalar density and $\bar{\bar{\Psi}} = \Psi$. Now *there is* a second definition that satisfies these requirements. We introduce the operator $\hat{W}$ that in momentum representation is a real function defined by

$$\hat{W}(p) = \theta(p_0) - \theta(-p_0) = \frac{p_0}{|p_0|} \tag{38}$$

with step function θ. Now, $\hat{W}$ is a pseudo-differential operator identical with the Hilbert transform with respect to p_0 (Friedrichs, 1970). $\hat{W}$ changes the relative sign between particle and antiparticle solutions. It is invariant under transformations of the proper Lorentz group and the parity transformation, since these do not change the sign of the timelike component p_0; this changes its sign only under time inversion and complex conjugation. We now define a new adjoint spinor by

$$\tilde{\Psi} = (\hat{W}\Psi)^{*T}\gamma^0 \tag{39}$$

Then $\tilde{\Psi}\Psi$ is an invariant real scalar density, $\tilde{\tilde{\Psi}} = \Psi$, since $\hat{W}^2 = 1$ (except for the unphysical point $p_0 = 0$), and, moreover, it does not change sign under the time inversion (35). As a result the action

$$\tilde{S}(\Psi) = \mathscr{R} \int d\tau \int d^4x\, \tilde{\Psi}(x, \tau)(i\hbar\partial_\tau + i\hbar c\gamma^\mu\partial_\mu)\Psi(x, \tau) \tag{40}$$

that yields the same equation of motion as $S(\Psi)$ is now invariant under a complex linear representation of the full Lorentz group and thus has the same invariance group of coordinate transformations as the classical action (1). Moreover, both $S(\Phi)$ and $\tilde{S}(\Psi)$ are invariant under the τ-inversion transformation $\mathscr{A}$ that extends the reparametrization transformation of the classical theory and is defined by

$$\begin{aligned} (\mathscr{A}\Phi)(x, \tau) &= \Phi^*(x, -\tau) \\ (\mathscr{A}\Psi)(x, \tau) &= i\gamma^2\Psi^*(x, -\tau) \end{aligned} \tag{41}$$

It changes the sign of all momenta. But as a reparametrization does not change the physical situation, the observed quantities may not change. This will be proven in Section 6. But then we obviously have a contradiction to the standard theory (Bjorken and Drell, 1964), where (41) is the charge conjugation transformation. Since the transformation (41) changes the sign of p_0, it is clear that one of the actions $S(\Psi)$ or $\tilde{S}(\Psi)$ cannot be invariant under this transformation. Indeed it is $S(\Psi)$ which is not and we now prove this explicitly, since it is a major point of criticism regarding the standard theory. We look at the mass term $m_0\bar{\Psi}\Psi$ of the standard action. For the transformed spinor we have

$$\overline{(i\gamma^2\Psi^*)}(i\gamma^2\Psi^*) = \Psi^T\gamma^{2*T}\gamma^0\gamma^2\Psi^* = \Psi^T\gamma^0\gamma^2\gamma^2\Psi^*$$
$$= -\Psi^T\gamma^0\Psi^* = -\bar{\Psi}\Psi$$

since γ^2 is self-adjoint, $\gamma^{2*T}\gamma^0 = \gamma^0\gamma^2$, and $\gamma^{0T} = \gamma^0$ in any representation. Hence the action $S(\Psi)$ is not invariant under the standard charge conjugation transformation. But if we instead employ the action $\tilde{S}(\Psi)$, we are in need of a charge conjugation transformation. For this purpose we give in Section 5 a five-dimensional formulation of relativistic quantum theory, since in the classical theory the behavior of charged particles is naturally studied in a five-dimensional formulation. Before we do this we note that the time inversion (35) commutes with the τ-inversion (41) and the parity transformation can be made to commute with it if we choose

$$x^{\mu\prime} = P^\mu{}_\nu x^\nu, \qquad p'_\mu = p_\nu P^{-1\nu}{}_\mu, \qquad P = \mathrm{diag}(1, -1, -1, -1)$$
$$\Psi'(x') = \pm i\gamma^0\Psi(x) \tag{42}$$

Hence $P^2 = -1$ for spinors. Moreover, it is obvious that in our theory the transformation PT is identical with the standard PCT transformation, and hence the PCT theorem (Pauli, 1955; Streater and Wightman, 1964) will hold in a corresponding field theory.

5. FIVE-DIMENSIONAL QUANTUM THEORY

The inclusion of gravitation and electromagnetism in a five-dimensional metric theory formulated by Kaluza (1921) and Klein (1926*a*) is extended to relativistic quantum theory by Klein (1926*b*) and Souriau (1963), where the four-dimensional theory is obtained as an approximation. The classical five-dimensional formulation given in Hannibal (1991) is a nonmetric theory, but equivalent to the four-dimensional one. We therefore now seek a formulation where the five-dimensional equations and actions exactly reduce to the four-dimensional equations for separating solutions. The five-dimensional space-time is assumed to be $\mathbb{R}^4 \times S^1$; the fifth coordinate x^4 ranges from 0 to 2π in units of $\hbar/e$, which gives to the fifth momentum p_4 directly the dimension of charge. The charge is first quantized by

$$q \to \hat{p}_4 = i\hbar \frac{\partial}{\partial x^4} \tag{43}$$

We use Latin indices $a, b, \ldots,$ for the range from 0 to 4; Greek indices still run from 0 to 3. The replacement

$$p_\mu \rightarrow p_\mu - p_4 A_\mu(x) \tag{44}$$

is easily done for the generalized Dirac equation (5). We define five matrices χ^a by

$$\chi^\mu = \gamma^\mu, \qquad \chi^4 = -\gamma^\mu A_\mu(x) \tag{45}$$

and write down the action as

$$\tilde{S}(\Psi) = \mathscr{R} \int d\tau \int d^5x\, \bar{\Psi}(x, \tau)(i\hbar\partial_\tau + i\hbar c\chi^a\partial_a)\Psi(x, \tau) \tag{46}$$

This is one action for spin-1/2 particles of any mass or charge, just as in the classical case. For a state of definite charge $q = n \cdot e$ the ansatz

$$\Psi(x^a, \tau) = \left(\frac{e}{2\pi\hbar}\right) e^{iqx^4/\hbar}\Psi'(x^\mu, \tau) \tag{47}$$

gives back the action (37). The quantization of charge into multiples of the elementary charge e is induced by the topology of the fifth dimension, since we require wave functions to be continuous. The action (46) is superficially five-covariant, but not truly, since the correct transformation properties of the χ-matrices do not follow from those of the γ-matrices; we still have to prescribe in a non-five-covariant way how the vector A_μ transforms. In the classical theory (Hannibal, 1991), $-A^\mu$ transforms a component of the singular 5×5 matrix $\tilde{g}^{ab}$

$$\tilde{g}^{ab} = \begin{bmatrix} g^{\mu\nu} & -A^\mu \\ -A^\nu & A^\kappa A_\kappa \end{bmatrix} \tag{48}$$

and indeed we see that without gravitation we have

$$\chi^a\chi^b + \chi^b\chi^a = 2\tilde{g}^{ab} \tag{49}$$

in extension of the relations for the γ-matrices, but we do not see how the theory may be formulated covariantly with the help of this matrix. In the spin-0 case we could try to formulate an explicitly five-covariant action with a pseudo-differential mass operator $\hat{M}$ defined by

$$(\hat{M}\Phi)(x, \tau) = \int \frac{d^5p}{(2\pi\hbar)^5}\, e^{-ip_\mu x^\mu/\hbar} c^{-1}[p_a\tilde{g}^{ab}(x)p_b]^{1/2}\Phi(p, \tau) \tag{50}$$

where the integration over the fifth momentum reduces to a sum over the charge eigenfunctions. The corresponding action is given by

$$S(\Phi) = \mathscr{R} \int_{\tau_0}^{\tau_1} d\tau \int d^5x\, \Phi^*(x, \tau)(i\hbar\partial_\tau + c^2\hat{M})\Phi(x, \tau) \tag{51}$$

But this concept is not satisfying, since $\hat{M}$ is not self-adjoint and we do not have $\hat{M}^2 = \tilde{g}^{ab}(x)\partial_a \partial_b$ in order that the solutions of the equation of motion satisfy the generalized Klein–Gordon equation (6) with the replacement (44) in the case of an electromagnetic potential with Lorentz gauge. Hence we restrict ourselves to spin 1/2. In this five-dimensional formulation based on the action (46) the charge conjugation becomes a coordinate transformation given by

$$\begin{aligned} x^{\mu\prime} &= x^\mu, \qquad x^{4\prime} = 2\pi\hbar/e - x^4 \\ p_a{}' &= p_b(C^{-1})^b{}_a, \qquad C = \mathrm{diag}(1, 1, 1, 1, -1) \\ \tilde{g}^{ab}(x') &= C^a{}_c C^b{}_d \tilde{g}^{cd}(x) \\ \Psi'(x') &= \Psi(x), \qquad \Phi'(x') = \Phi(x) \end{aligned} \tag{52}$$

The transformation of A_μ is induced by the transformation of $\tilde{g}^{ab}$. For uncharged particles the transformation reduces to the identity. The transformation is complex linear and leaves the action (46) invariant. As a result the complete symmetry groups of the five-dimensional classical and quantum actions are identical. $U(1)$ gauge transformations are induced by coordinate transformations $x^{4\prime} = x^4 + f(x^\mu)$, as can be seen from (47). But we note that this charge conjugation is not a symmetry realized in nature, since particles with opposite charge are not observed. The charge conjugation does not transform particles into antiparticles; this is achieved only by the time inversion of our theory.

6. DENSITIES AND CURRENTS

We now look at the currents that arise from the actions $S(\Phi)$ and $\tilde{S}(\Psi)$. In a laboratory a particle is observed irrespective of its proper time. Thus, in order to obtain observable currents we have to integrate over all contributing proper time. We also integrate over x^4 assuming that this coordinate is invisible as in the classical case (Hannibal, 1991). Hence the four-current that belongs to any momentum operator $\hat{B}$ (energy, momentum, charge) is given by

$$J_B{}^\mu = \mathscr{R} \int d\tau \int dx^4\, \Phi^* \hat{B} \hat{v}^\mu \Phi \tag{53}$$

for spin 0, with velocity operators,

$$\hat{v}^{\mu} = (c^2/i\hbar)[\hat{M}, \hat{x}^{\mu}] = \eta^{\mu\nu}\hat{p}_{\nu}\hat{M}^{-1} \tag{54}$$

and

$$J_{\hat{B}}{}^{\mu} = \mathscr{R}\int d\tau \int dx^4\, \tilde{\Psi}\hat{B}\hat{v}^{\mu}\Psi \tag{55}$$

for spin 1/2 with velocity operators

$$\hat{v}^{\mu} = c[\chi^a\partial_a, \hat{x}^{\mu}] = c\chi^{\mu} \tag{56}$$

Now we see that due to the definition of the adjoint spinor $\tilde{\Psi}$ the energy density $J^0_{p_0}$ is positive for all solutions of the generalized equations of motion; the operator $\hat{W}$ has changed the sign of the energy of antiparticle spinor solutions. On the other hand, the electric charge density $J^0_{p_4}$ now takes different signs for particle and antiparticle solutions for the same values of p_4. This is the same for scalar and spinor functions in accordance with the classical five-dimensional theory, where we showed that the observable charge is p_4 for particles and $-p_4$ for antiparticles (Hannibal, 1991). This means that we have to choose one and the same sign of p_4 for all solutions of the wave equations. These form two equivalent sectors which can be seen from the τ-inversion which changes the sign of p_4 and hence interchanges these sectors; but it may be seen that the τ-inversion leaves both currents (53) and (55) invariant for any operator that is odd in momentum representation, hence especially the energy-momentum tensors $J^{\nu}_{p_\mu}$ are invariant. This means that we do not need Dirac's (1929, 1930) hole theory for antiparticles; these can be treated just like particles and energy conservation will prevent them from decaying into particles and photons. But what about the probability density? We see that the probability or particle density, if conventionally defined as $J_1{}^0$, becomes negative now also for antiparticle spinor solutions, just as it already was for scalar antiparticle solutions, which is the basic problem in the interpretation of the Klein–Gordon theory. Moreover, it is not invariant under the reparametrization transformation $\mathscr{A}$ as any physical quantity has to be. The reason for this is that the operator 1 is even. To solve this problem, we suggest that the probability current is defined to be $J_{\hat{W}}{}^{\mu}$ which is invariant since $\hat{W}$ is odd. The density $J_{\hat{W}}{}^0$ is positive for all solutions of the equations (5) and (14), since it is the original density of the Dirac theory and the sign is changed for antiparticle solutions of the Klein–Gordon theory. If it may be normed to $\int d^3x\, J_{\hat{W}}{}^0 = 1$, it may be interpreted as a probability density. We therefore look at the conservation properties of the current $J_{\hat{W}}$. For scalar wave functions of free particles we have

$$\partial_\mu J_{\hat{W}}{}^\mu = -\tfrac{1}{2}\Phi^* \hat{W}\Phi \,|_{\tau_0}^{\tau_1} + \tfrac{1}{2}\mathscr{R}(\partial_\mu \Phi^*)\hat{W}(\hat{v}^\mu \Phi) \tag{57}$$

The rhs of (57) is zero if Φ is an eigenfunction of the mass operator. In this case the probability current is conserved and we may interpret Φ as a single-particle state. Otherwise, for a linear combination of overlapping eigenfunctions with different masses we get an oscillating part. For spinors we have

$$\partial_\mu J_{\hat{W}}{}^\mu = -\tilde{\Psi}\hat{W}\Psi \,|_{\tau_0}^{\tau_1} \tag{58}$$

which again yields the conservation of probability only if Ψ is an eigenstate to the mass operator. This generalization is important in view of interactions that change mass or charge and particles are created or annihilated. If we restrict our theory to free particles of definite mass, we see that we have constructed a relativistic invariant theory where the particle density and energy density are positive for all solutions of the equations of motion for spin 0 and spin 1/2. This explicitly disproves the statement given by Pauli (1940) that due to relativistic invariance such a theory does not exist. But we note that the proof of the spin-statistics theorem (Pauli, 1940; Streater and Wightman, 1964) in the framework of a corresponding field theory will not be influenced by the introduction of the operator $\hat{W}$ into the action.

The boundary term arising in (58) and the fact that a term

$$0 = \int dx^4 \, \partial_4(\tilde{\Psi}\hat{v}^4\Psi) \tag{59}$$

canceled in the calculation due to periodic boundary conditions suggest that we consider the five-dimensional τ- and x^4-dependent current

$$J_{\hat{B}}{}^a(\tau) = \mathscr{R}\tilde{\Psi}\hat{B}\hat{v}^a\Psi \tag{60}$$

since we then have the conservation property

$$\partial_\tau \tilde{\Psi}\hat{W}\Psi + \partial_a J_{\hat{W}}{}^a(\tau) = 0 \tag{61}$$

The trace of the energy-momentum five-tensor $J^b_{p_a}(\tau)$ is the mass density $\tilde{\Psi}\hat{M}\Psi$. This feature, which generalizes the property that the energy-momentum tensor of the electromagnetic field is trace-free since photons are massless, was not present in the four-dimensional theory. Then if the energy-momentum five-tensor $J^b_{p_a}(\tau)$ acts as a source of the external fields, for nonoverlapping charge and mass eigenfunctions it does not depend on x^4 or τ and the fields will be uncharged and massless. But if eigenfunctions with different charge or mass dependence overlap, the fields will be charged or massive. This shows that a five-dimensional theory of the fields with first quantized mass and charge *a priori* has to deal with charged and massive interactions that are necessarily short-ranged since they exist only for overlapping wave functions. Thus, at least the weak interaction should be included in any considerations for a unified theory in extension of the ideas of Kaluza and Klein. A new ansatz for such a theory will be presented elsewhere.

7. CONCLUSION

We conclude that the introduction of the proper time and a fifth coordinate into the wave functions together with the first quantization of mass and charge, and the inclusion of the sign operator $\hat{W}$ into the definition of the adjoint spinor and the probability current, have interesting consequences for relativistic quantum theory. First of all the theory is in every respect consistent with the classical theory; the equations of motion are related by the Ehrenfest theorem and all actions have the same symmetry group, including a complex linear representation of the full Lorentz group for the quantum actions. Then the two basic problems in the interpretation of relativistic quantum theory, the negative probability density of the Klein–Gordon theory and the negative energy states of the Dirac theory, are removed; the energy and probability densities for the solutions of the generalized equations are always positive. These features are retained if one specializes the wave functions to be eigenfunctions of the mass and charge operators, in which case we obtained the original equations of motion. Thus, particles and antiparticles may be treated in the same way as different solutions of some wave equation, in the absence of interactions we have a consistent single-particle interpretation.

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